Electrical Circuit Analysis: Notes

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Chapter 1: Basic Concepts

An **electric circuit** is defined as an interconnection of electrical elements. **Charge** is defined as an electrical property of the atomic particles which make up matter, measured in coulombs (C). The electron is defined to have a negative charge of $1.602 \times 10^{18} C$, while the proton's charge is positive with the same magnitude. The *law of conservation of charge* states that charge can neither be created nor destroyed, only transferred. Meaning the algebraic sum of charges in a system does not change.

A unique feature of charges is that they can move. Conventionally we consider the movement of positive charges. **Electric current** is defined as the rate of change of charge with respect to time, measured in amperes (A). Mathematically that is:

$$\boxed{i(t) = \frac{dq}{dt}}$$

...and the amount of charge transferred between time t_0 and t is:

$$\boxed{Q = \int_{t_0}^{t} i(t)\ dt}$$

Direct current flows only in one direction and can be constant or time varying. **Alternating current** is a current that changes direction with respect to time. By convention I represents constant current while i represents a current that varies with time. Current flow can have a positive or negative value indicating relative directions.

In order for a charge to move through a conductor, a force must act on it. This force is known as voltage or potential difference, and is always measured between to points. Mathematically it is defined as:

$$V = \frac{dw}{dq}$$

...where w is the energy in joules (J) and q is the charge. Therefore **voltage** is defined as the energy required to move a unit charge from one reference point to another measured in volts (V). Notably:

$$V_{ab} = -V_{ba}$$

Voltage is the energy absorbed or expended as a unit charge moves through a circuit element, and is created by a separation of charge. The polarity of voltage is used to indicate direction.

Power is defined as the rate of change of energy with respect to time, as in:

$$\boxed{p = \frac{dw}{dt} = \frac{dw}{dq} \times \frac{dq}{dt} = \pm vi}$$

A circuit element that absorbs power has a positive p value, while a circuit element that produces power has a negative p value. In order to understand which of $\pm$ to use, you must understand passive sign convention.

Current through a passive element always goes into the positive terminal and out through the negative terminal. For passive convention: $p = vi$, for non-passive convention $p = -vi$. A notable fact about power within a circuit is that:

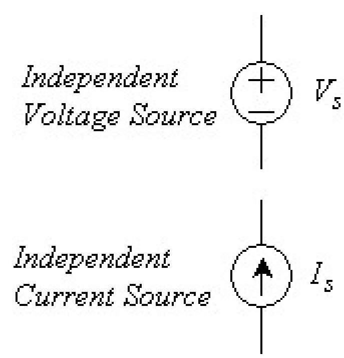
$$\sum p = 0$$

Furthermore, as a consequence of the $p = vi$ relationship:

$$w = \int_{t_0}^{t} v(t)\ i(t)\ dt$$

Sources are two terminal elements that supply energy to the circuit. A source can either be independent or dependant. First we will focus on **independent sources**.

An **independent voltage source** is an active device that provides a specified voltage no matter the current. Similarly, an **independent current source** is an active device that provides a specified current no matter the voltage across its terminals. The following are the symbols for each:



Ideal components are independent of resistance. During this course, we assume all sources are ideal.

Chapter 2: Basic Laws

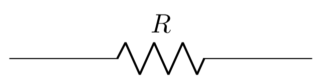
Resistors are elements of a circuit that have the property of resistivity, which is defined as the ability of a material to resist the flow of charge. The value of resistance depends on the material property, length, and surface area of a cross-section, as in:

$$R = \rho \frac{l}{A}$$

Resistors have a maximum wattage that they can handle. Resistance is measured in Ω. While every material resists the flow of current, there are two notable types of materials:

1. **Conductors** which have negligible resistance ($< 0.1\ \Omega$)
2. **Insulators** which have very high resistance ($> 50\ \mathrm{M}\Omega$)

The following symbol is used to denote a resistor:



Ohm's law states that the voltage across a resistor is directly proportional to the current flowing through it. Mathematically that is:

$$\boxed{v = \pm iR}$$

...where the sign depends on the passive sign convention. Voltage and current have no relation over an independent power source, but over a resistor it is related by Ohm's law. Other notable equations derived from Ohm's law include:

$$P = \frac{v^2}{R}$$

$$P = Ri^2$$

Since an R value can range from 0 to ∞, it is important to know what happens at the extremes. A resistance of 0 is known as a **short circuit**, which would make the voltage $= 0$ and the current i tend to ∞.

Similarly, an element with an R value of ∞ is known as an **open circuit**, which means the current is zero, and the voltage could take any value.

It is sometime more logical to consider **conductance** which is the inverse of resistivity as in it is the measure of the ability of an element to conduct electric current. Conductance is measured in siemens (S) or mhos. Notably:

$$G = \frac{1}{R} = \frac{i}{V}$$

$$i = \pm Gv$$

$$p = v^2 G = \frac{i^2}{G}$$

Any resistors that obeys Ohm's law are considered *linear* resistors, otherwise they are *nonlinear*.

Next we will discuss the building blocks of a circuit. First of all there are **elements** which are items in the circuit. Each element has a certain number of **terminals**, where a terminal is an electrode by which electricity can flow. A single two-terminal element in a circuit is called a **branch**. This does not include segments of wire.

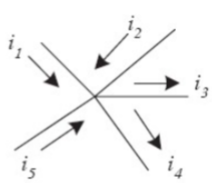
A **node** is a point of connection between two or more branches in a circuit. This could include an entire portion of a circuit. An *essential node* is the point of connection between three or more branches. A node is typically denoted by a dot in a diagram. However, if a short circuit is between two nodes, they count as one. So a node directly connected to another node with no element between them is considered one node. A **loop** is defined as any closed path on a circuit, while **mesh** is defined as the set of the smallest closed paths in a circuit. A loop is said to be *independent* if it contains at least one branch which is not part of any other independent loop.

A network with b branches, n nodes and l loops will satisfy the *fundamental theorem of network topology*:

$$b = l + n - 1$$

Two or more elements are in **series** if they exclusively share a single node and consequently carry the same current. Multiple elements in series can be rearranged and the circuit is equivalent. Two or more elements are in **parallel** if they are connected to the same two nodes and consequently have the same voltage across them.

Kirchhoff's Current Law states that the algebraic sum of currents entering a node is zero, which practically means that the sum of currents entering a node must equal the sum of the currents leaving the node. This is based on the law of conservation or charge.



$$\begin{aligned} i_1 + i_2 - i_3 - i_4 + i_5 &= 0 \\ i_1 + i_2 + i_5 &= i_3 + i_4 \end{aligned}$$

Mathematically that is:

$$\boxed{\sum_{n=1}^{N} i_n = 0}$$

... where N is the number of branches connected to the node.

Kirchhoff's Voltage Law states that the algebraic sum of voltages around a closed path is zero. If you go around a closed loop and calculate all the voltages across each component, then they will add to zero, taking into account passive convention.

Mathematically that is:

$$\boxed{\sum V_m = 0}$$

Resistor Laws

For any two resistors, there is a resistor called R_{eq} that if put in place of the two resistors, would have the same effect.

For two resistors in series:

$$\boxed{R_{eq} = \sum R}$$

This is because the two resistors in series must carry the same current.

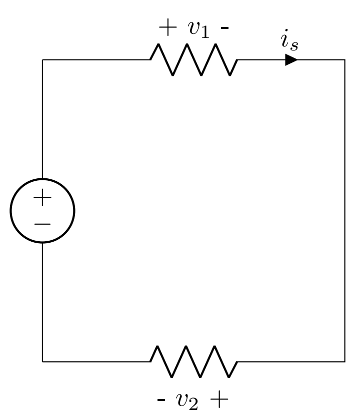
For two resistors in parallel:

$$\boxed{R_{eq} = \left(\sum \frac{1}{R}\right)^{-1}}$$

In particular for two resistors that is:

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

Voltage Divider



The voltage across resistors R_1 and R_2 are as follows:

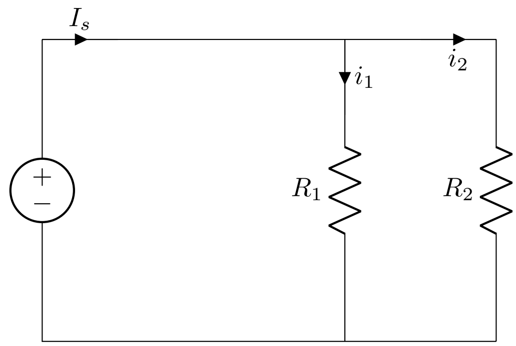
$$v_1 = \frac{R_1}{R_1 + R_2} v_s$$

$$v_2 = \frac{R_2}{R_1 + R_2} v_s$$

And in general, if a voltage divider has N resistors in series with source voltage v_s, the nth resistor will have a voltage drop of:

$$\boxed{v_n = \frac{R_n}{\sum_{n=1}^{N} R_n} v_s}$$

Current Divider



In a situation like the circuit above, the following applies:

$$\boxed{i_1 = I_s \frac{R_2}{R_1 + R_2}}$$

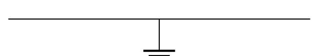
$$\boxed{i_2 = I_s \frac{R_1}{R_1 + R_2}}$$

Chapter 3: Analysis Methods

Nodal Analysis

Nodal analysis is a method used to identify voltage data in a circuit. Essentially you create n equations relating to n voltage variables and solve for each voltage. To begin, we must understand what is meant by voltage at a point. One point in the circuit is chosen as the *reference* or *ground*. This allows is to calculate "voltage at a point" by actually getting the voltage between that point and the ground. The value of the voltage at the ground is not important, and typically set as 0.

Importantly, the voltage drop from point A to point B is equal to $v_a - v_b$. The voltage drop across a resistor is $v_+ - v_-$. The symbol for ground is:



There are 7 steps to **Nodal Analysis**:

1. Remove the variable names (clean up the image). This step is not necessary but will help reduce any confusion.

2. Label all essential nodes.

3. Pick an essential node as reference. Set its voltage to be 0.

4. Label the voltages of the remaining essential nodes.

5. Do KCL at each node, and assume all current is leaving in all directions from each node. Use the following to help find the currents:

$$i = \frac{v_1 - v_2}{R}$$

6. You will now have n equations and n unknowns. Solve.
7. Use the values of the voltages at each essential node to solve for missing information.

If you have a *voltage source in your circuit* that is not connected to the reference node, you must create a **supernode** that encompasses the two nodes on either side of the voltage source and the voltage source itself, and consider it to be one super node. You gain an equation by doing this in the form of: the voltage difference across the voltage source is equal to the value of the voltage source (following passive convention). You can then do KCL at that supernode (still considering the smaller nodes within while doing it). Remember a voltage source always immediately has a node on either side of it. If the voltage source is between the reference node and a non-reference node, then you can just assign the value of the voltage source to the non-reference node (following passive convention) and no super node is needed.

Dependant Sources

A dependant source is a source who's value depends on another part of the circuit. Typically written in a diamond shape instead of a circle with the value shown next to it (value contains a variable). Convert all the units to standard SI and then work from there, the units don't matter when solving, just assign the correct ones.

Mesh Analysis

Mesh analysis is useful for finding all the currents through the circuit. Recall: a mesh is the smallest loop in a circuit. There are commonly more than 1 mesh in a circuit. Meshes do not need to be visually the same size, the just need to not overlap any other meshes. Through each mesh, you create an imaginary current usually denoted I_n for the n^{th} mesh. You then do KVL through each mesh and write all the voltages in terms of the I_ns, as in $V = iR$. Note that the current going through an element that connects two meshes will have a current of $I_a - I_b$ when doing KVL from the perspective of mesh A.

Steps to mesh analysis:

1. Clean up the circuit.
2. Label each mesh with imaginary current I_n.
3. Do KVL at each mesh writing the voltages as:

$$v = \pm R(I_a - I_b)$$

4. Solve for each I_n (n equations and n unknowns).

5. Deduce any current in the circuit from the set of I_ns.

If you have a *current source in your circuit*, which is touching more than one of the $I_n s$ then you need to create a **supermesh** which contains the two meshes and the current source. You gain an equation by being able to show the relationship between the two $I_n s$ in relation to the current source. You can then do KVL around the supermesh. If the current source is on the border of the circuit, then you can simply assign the value of the related I_n to that current source (following the direction of the currents) and then don't do KVL on that mesh, no supermesh needed.

Chapter 4: Circuit Theorems

Superposition Method

A *linear circuit* is one whose output is linearly related to its input, for example any basic circuit with resistors and voltage/current sources where *Ohm's law* holds.

The method of **superposition** can be used on linear circuits. Superposition is a broad concept used in many fields, relating to linear circuits it means that each voltage/current source has an independent effect on some other element of a circuit (for example the voltage across some resistor), and that *the sum of those independent effects* results in the actual value you are looking for.

"n people pushing on a box will be as good as the sum of how far each person can push the box alone."

Steps for Superposition:

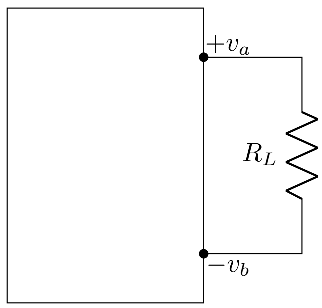
1. Turn off all but one independent source, and leave dependent sources.
   - For voltage sources replace them with a short circuit.
   - For current sources replace them with an open circuit.

2. One at a time, for each independent source in the original circuit, calculate the intermediate value of what you are looking for, and denote it with a prime.

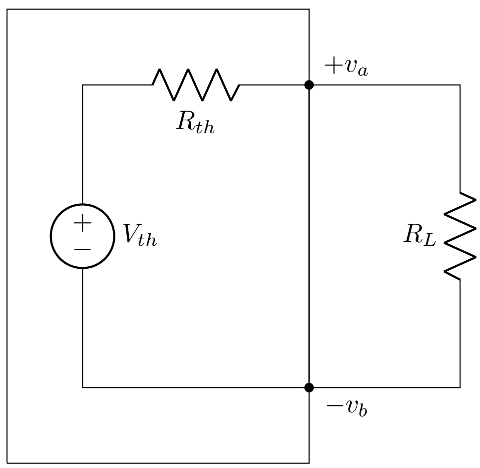
3. Take the sum of all the intermediate values, this result is equal to the value of what you are looking for.

Thevenin's Theorem

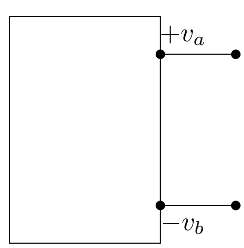
A **Thevenin equivalent circuit** is a two terminal circuit that can replace another more complicated circuit. The *Thevenin* circuit is made up of a single voltage source with value v_{th}, and a single resistor in series with a resistance of R_{th}. That *Thevenin* circuit must be connected to a variable load called R_L.



... where the empty box represents a linear circuit, is equivalent to:



The goal in general is to find the values of V_{th} and R_{th} that make this true for two given nodes v_a and v_b. In order to find V_{th}, you simply use the circuit in the box and solve for $V_{th} = v_a - v_b$, and replacing R_L with an open circuit, as in:



To solve for R_{th} you must turn off all independent sources, and reduce the resistors in the boxed circuit to just one resistor, which will have a value of R_{th}. However, if there are dependant sources this is not possible, in this case you must:

- Replace R_L with a voltage source with an arbitrary value V_{test}

- Turn off all independent sources.
- Solve for the current through the voltage source, I_{test}
- Solve for R_{th} by using:

$$R_{th} = \frac{V_{test}}{I_{test}}$$

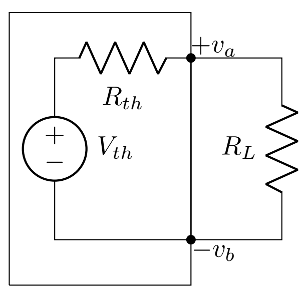
Maximum Power Transfer

Suppose that $R_L = 0$, then $V_L = 0$, now suppose $R_L = \infty$, then $i_L = 0$. In both these cases, the *power* delivered to the resistor $R_L = 0$ since:

$$p = vi$$

So for what voltage between $0 \to \infty$ is the power delivered to the resistor R_L at it's maximum?

Suppose you have the following circuit:



... then the power over the resistor is:

$$p = v_L i_L = \left(\frac{R_L}{R_L + R_{th}}\right) V_{th} \cdot \frac{V_{th}}{R_L + R_{th}} = V_{th}^2 \cdot \frac{R_L}{(R_L + R_{th})^2}$$

We want to maximize p, to do this we use the following:

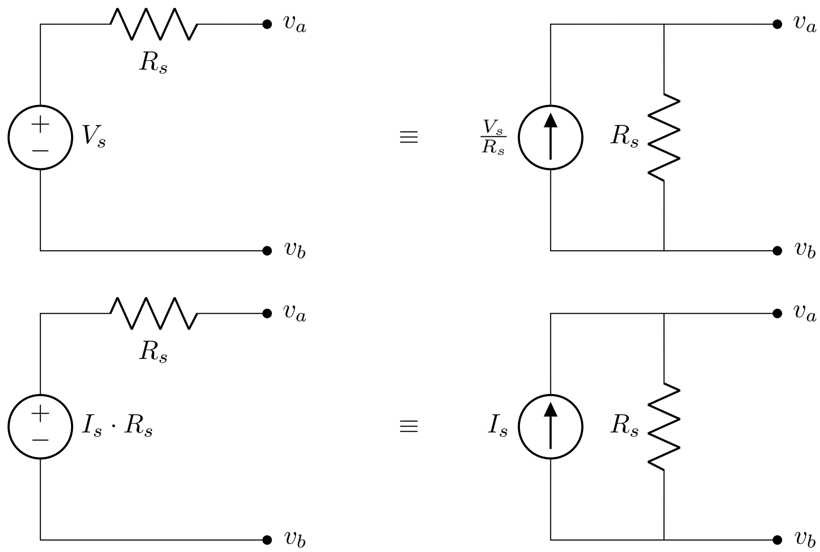
$$\frac{dp_L}{dR_L} = 0 \implies R_{th} = R_L$$

... keeping in mind that both R_{th} and V_{th} are constant. The maximum power (p_{max}) delivered to the load resistor R_L occurs when:

$$\boxed{R_L = R_{th}} \quad \text{then...} \quad \boxed{p_{max} = \frac{V_{th}^2}{4R_{th}}}$$

Source Transformations

It is possible to transform a voltage source into a current source, and vice versa, using the method of *Source Transformations*. The following pairs of circuits are equivalent:



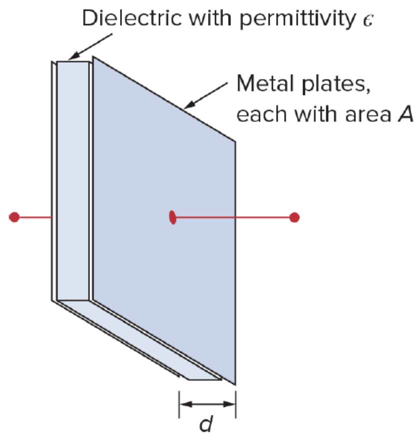
Norton's Theorem

Norton's theorem is very similar to Thevenin's and the Thevenin equivalent circuit. The Norton's equivalent circuit contains a single current source and resistor in parallel. This is effectively just a Thevenin equivalent circuit that had been source transformed. You can find the Norton equivalent circuit by first finding R_{th} (which can be called R_N but they are equivalent) and then assigning that value to R_L and solving for the current through R_L from node a to node b.

Chapter 6 & 7: Capacitors and Inductors

Capacitors

A **capacitor** is a two-terminal passive element designed to store and release energy in its electric field. Both capacitors and inductors *depend on time*. Physically, the capacitor is made of two conductive plates separated by a strong insulator (also called *dielectric*) of width d. One metal plate builds up a strong negative charge, which causes a net positive charge to form on the other plate. This separation of charge causes a voltage difference.



An important measure of a capacitor is the *capacitance* very similar to *resistivity* in resistors. Capacitance is the ratio of the charge on one plate to the voltage difference between the two plates. Capacitance is defined as:

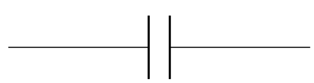
$$\boxed{C = \frac{\varepsilon A}{d}}$$

Where:

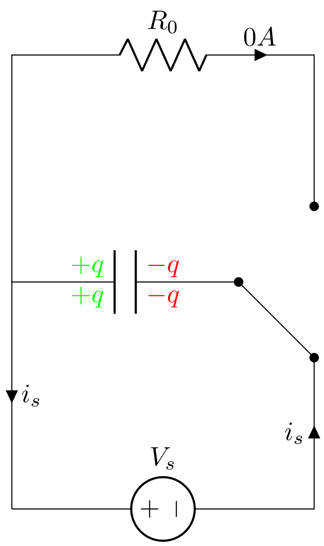
- ε is the permittivity of the insulator.
- A is the surface area of each plate.
- d is the distance between plates.
- C is the capacitance measured in *Farad* ...

$$1F = \frac{1C}{1V}$$

The following circuits will aid in describing the purpose and function of a capacitor. Note first that the symbol used for a capacitor is:



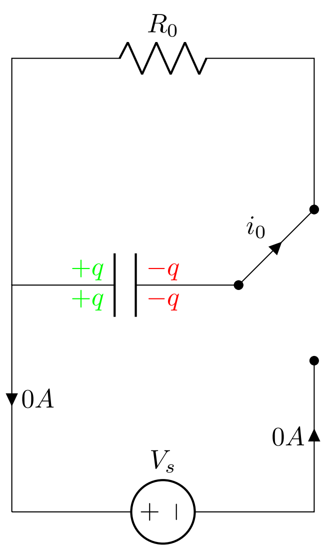
Say you have a circuit like the following, where the junction on the right is a switch that can alternate between a raised and lowered position:



Evidently, there is no current going up to the resistor, but there is a (non-standard) current going through the bottom mesh even though current is not passing through the capacitor due to the *dielectric*. There is a charge building up on either side of the capacitor. The value of that charge is defined using the following equation:

$$\boxed{q = CV_s}$$

This process will continue until the voltage across the capacitor is equal to V_s in which case $i_s = 0$ and the circuit "stops". Once this happens, we flip the switch, and the circuit would act as follows:



As you can see now, the current through the original voltage source is $i_s = 0$, and there is a new current i_0 which is created by the capacitor. The capacitor gradually releases its charge, and thus depletes its voltage difference. The following is the voltage to current relationship over a capacitor, similar to how we used Ohm's Law with resistors:

$$\boxed{i(t) = C\frac{dv}{dt}}$$

$$\boxed{v(t) = \frac{1}{C}\int_{t_0}^{t} i(t)\ dt + v(t_0)}$$

The energy stored in the capacitor is therefore:

$$\boxed{w = \frac{1}{2}Cv^2 = \frac{q^2}{2C}}$$

There are four main important properties of capacitors:

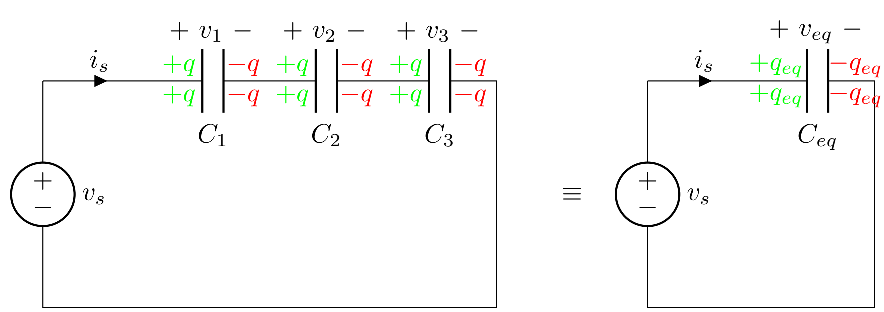
1. A capacitor is an open circuit to DC. Since $v(t) = V_s$ and V_s is a constant implying that $\frac{dv}{dt} = 0$.
2. The voltage on a capacitor cannot change abruptly. The flow of electrons caused by the capacitor can never be instantaneously released or stopped, it is always gradual. This means that the graph of voltage versus time of a capacitor must be continuous. We can then use the definition of continuity which is that, for any time t_0, the following is true:

$$v(t_0^-) = v(t_0^+)$$

3. An idea capacitor does not dissipate energy.
4. A real capacitor has *parallel-model leakage resistance* which can be neglected for most applications, we assume ideal capacitors.

Capacitors in Series and Parallel

Suppose you have the following two equivalent circuits:



Naturally we want to know what are the values of C_{eq}, v_{eq}, and q_{eq}.

First let's figure out v_{eq}, by doing KVL on both circuits we get the two equations:

$$-v + v_1 + v_2 + v_3 = 0 \implies v = v_1 + v_2 + v_3$$

$$-v + v_{eq} = 0 \implies v = v_{eq}$$

Which tells us that:

$$v_{eq} = v_1 + v_2 + v_3$$

So for n capacitors in series:

$$\boxed{v_{eq} = \sum_n v_n}$$

Next let's discuss q_{eq}. The charge carried over each of the capacitors is equal, given by:

$$q = Cv$$

This means that Cv is constant, and so:

$$q_1 = q_2 = q_3 = q_{eq}$$

So for n capacitors in series:

$$\boxed{q_{eq} = q_n}$$

Lastly let's figure out C_{eq} from the KVL equation we used to find v_{eq}:

$$v_{eq} = v_1 + v_2 + v_3$$

... recall that:

$$q = Cv \implies v = \frac{q}{C}$$

... so the v_{eq} equation becomes:

$$\frac{q_{eq}}{C_{eq}} = \frac{q_1}{C_1} + \frac{q_2}{C_2} + \frac{q_3}{C_3}$$

$$\frac{q_{eq}}{C_{eq}} = \frac{q_{eq}}{C_1} + \frac{q_{eq}}{C_2} + \frac{q_{eq}}{C_3}$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

So for n capacitors in series:

$$\boxed{C_{eq} = \left(\sum_n \frac{1}{C_n}\right)^{-1}}$$

Similar reasoning can be used to derive the following equations for **capacitors in parallel**:

$$\boxed{v_{eq} = v_n}$$

$$\boxed{q_{eq} = \sum_n q_n}$$

$$\boxed{C_{eq} = \sum_n C_n}$$

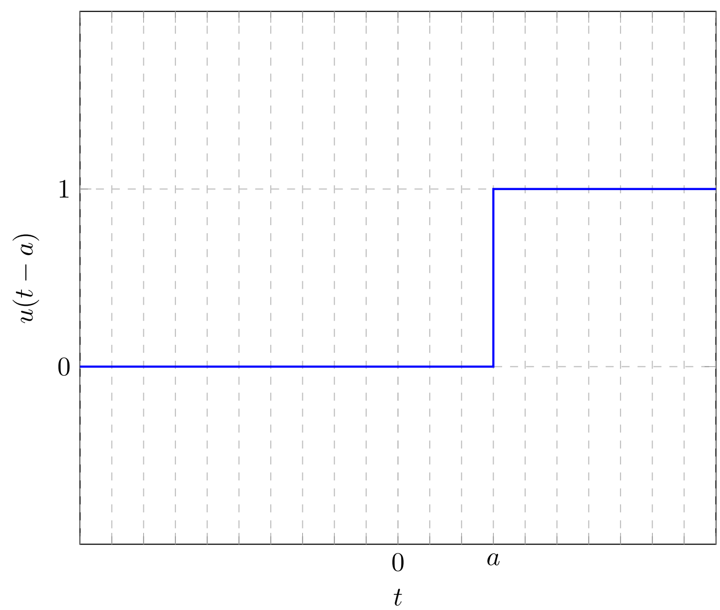
Unit Step Function

The **unit step function** is a function defined as:

$$u(t-a) \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

... and it's graph is as follows:

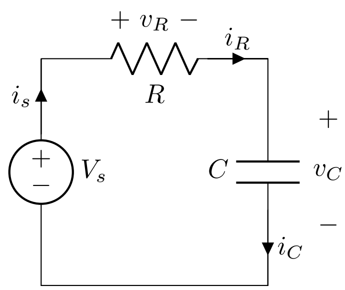
The Graph of the Unit Step Function



This function is used to describe some behaviour in the next sections of circuits. Essentially, at time $t = a$ the function jumps from a value of 0 to 1.

Response of First Order RC Circuit

An **RC Circuit** is a circuit which contains resistors and capacitors. The basic example is the following:



We want to be able to determine v_C and i_C. The first thing to note is that:

$$i_s = i_R = i_C$$

.. and that:

$$i_C = C\frac{dv}{dt}$$

... using *Ohm's Law* on the resistor:

$$v_R = i_C R \implies v_R = c\frac{dv}{dt}R$$

We can then do a single KVL loop around the circuit above, giving us:

$$-V_s + v_R + v_C = 0$$

$$-V_s + RC\frac{dv}{dt} + v_C = 0$$

This is a first order differential equation in $v_C(t)$, and it's solution is:

$$v_C(t) = V_s + \left[(v_C(t_0) - V_s)e^{\frac{-(t-t_0)}{RC}}\right]$$

... which is also commonly written in the form:

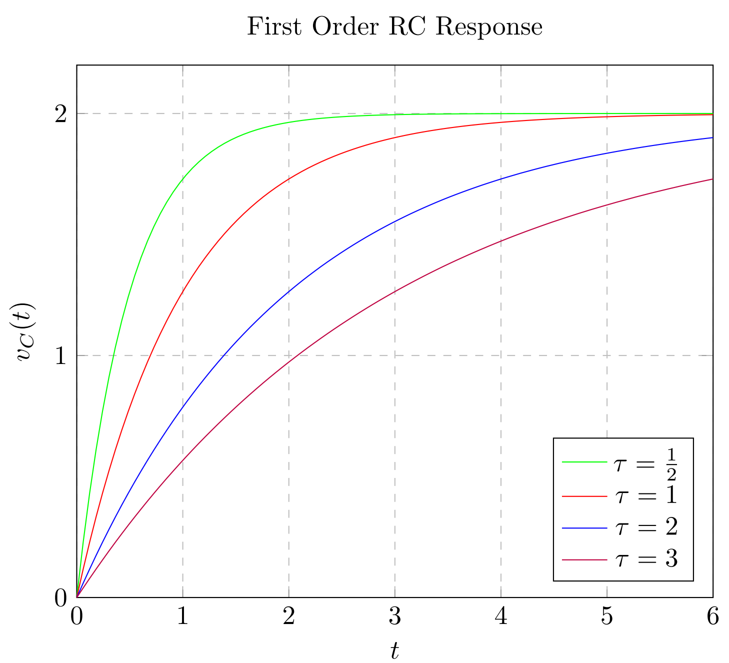
$$\boxed{v_C(t) = V_s\left(1 - e^{\frac{-(t-t_0)}{RC}}\right) + v_C(t_0)e^{\frac{-(t-t_0)}{RC}}}$$

The first term in this second form is called the *input excitation* and is the response from external sources. The second term is called the *natural/stored charge* response, as it contains the voltage at time $t = t_0$.

If $t_0 = 0$ and $v_C(t_0) = 0$ then the equation simplifies to:

$$v_C(t) = V_s\left(1 - e^{\frac{-(t-t_0)}{RC}}\right)$$

The response can be graphed for some set values ($V_s = 2$, $t_0 = 0$, $RC = 1$, $v_C(t_0) = 0$), one thing to note is that the product RC us usually called τ the time constant, and is a measure of the amount of time it takes to get to the steady state response (which will be defined after the graph). The following graph contains multiple plots varying only in τ:



Notice that varying the time constant only changes the rate at which the graph changes, and does not change the initial or final values of the voltage. The plots evidently have two different response phases, the first phase called the **transient response** occurs while the voltage is still increasing. The **steady state response** occurs when the voltage levels out to a specified value (namely V_s). It takes approximately 5τ units of time to get to steady state.

Finally note before we move on, another way to think about the solution to the differential equation in the form:

$$v_C(t) = V_s + \left[(v_C(t_0) - V_s)e^{\frac{-(t-t_0)}{RC}}\right]$$

... is (when $t_0 = 0$):

$$v_C(t) = v_C(\infty) + \left[v_C(0^+) - v_C(\infty)\right] e^{\frac{-t}{\tau}}$$

This says that $v_C(\infty) = V_s$ which is true because the steady state of the capacitor has a voltage value equal to the voltage source. This perspective is useful for solving RC circuits as described in the following section.

Based on this solution we can derive the formula for $i_C(t)$ using the fact that:

$$i_C(t) = C\frac{dv_C}{dt}$$

... which gives us:

$$\boxed{i_C(t) = \frac{V_s}{R_{eq}} e^{\frac{-t}{\tau}} \text{ for } t > 0}$$

Source-Free RC Response

The source free response of an RC circuit is only the natural/stored charge response, as there is no external sources to excite it. The following equation describes it:

$$v(t) = V_0 e^{\frac{-t}{\tau}}$$

... where V_0 is the capacitor's initial charge.

Steps to Solve a First Order RC Circuit ($\mathbf{t}_0 = 0$)

1. Draw the circuit with initial position of the switch. (Note that a switch is usually used to control the charge of the capacitor).

2. Calculate the voltage over the capacitor ($v_C(0^-)$), noting that when the capacitor is fully charged it acts as an open circuit.

3. $v_C(0^-) = v_C(0^+)$

4. Draw the circuit in the final position of the switch.

5. Calculate $v_C(\infty)$ which will occur when the capacitor is fully discharged assuming there are no more complications and the circuit contains only resistor and sources otherwise.

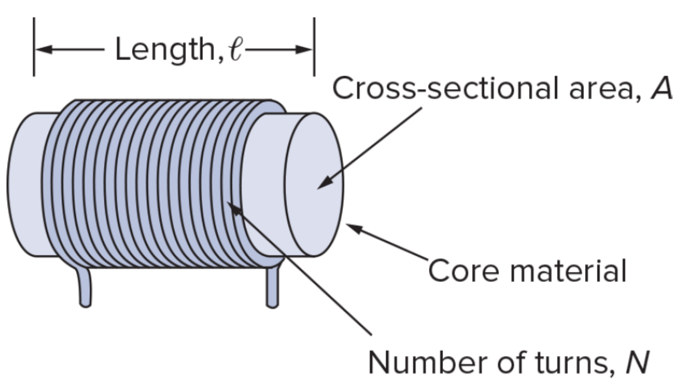
6. Calculate $\tau = R_{eq} \cdot C$, where R_{eq} is the total resistance seen from the perspective of the capacitor with all independent sources turned off.

7. Substitute all these values into the following equation:

$$v_C(t) = v_C(\infty) + \left[v_C(0^+) - v_C(\infty)\right] e^{\frac{-t}{\tau}}$$

Inductors

An inductor is very similar conceptually to a capacitor. Just as the capacitor stored energy in their electric fields, an inductor stores energy in it's magnetic field. In general it is made of a coil of conducting wire.



Inductance (L) is the property whereby an inductor exhibits opposition to the change of current flowing through it. The inductance of an inductor is measured in *Henrys* (H) where:

$$H = \frac{V_s}{A}$$

... L can be calculated by:

$$L = \frac{N^2 \mu A}{\ell}$$

Where:

- N is the number of turns in the coil.
- ℓ is the length of the coil.
- A is the cross-sectional area of the core material (which can be made of iron, steel, plastic, or air).
- μ is the permeability of the core material.

If current is allowed to pass through an inductor, the voltage can be modeled by:

$$v = L\frac{di}{dt}$$

... or equivalently:

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v_L(t)\ dt + i_L(t_0)$$

The symbol used for an inductor is:

A **linear inductor** is one who's inductance does not depend on current and thus is constant. The graph of the v vs $\frac{di}{dt}$ relationship is thus linear with a slope of L. Otherwise the inductor is considered to be **non-linear**.

To calculate the power through a inductor use:

$$p = v \cdot i \implies p_L = v_L \cdot i_L = \boxed{L\frac{di}{dt} \cdot i_L}$$

... and similarly for energy:

$$\boxed{w = \frac{1}{2}L[i_L(t) - i_L(t_0)]^2}$$

... and for $i_L(t_0) = 0$:

$$w = \frac{1}{2}Li^2$$

There are three important properties of inductors:

1. If the current through the inductor is constant, then $v_L = 0$. This means an inductor acts like a short circuit for DC.
2. Current flowing through the inductor cannot change instantaneously. It has an opposition to the change in current flowing through it.
3. An ideal inductor does not dissipate any energy.

Inductors in Series and Parallel

The equivalent inductance (L) for inductors in series is:

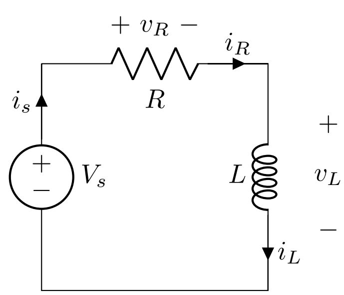
$$\boxed{L_{eq} = \sum_n L_n}$$

The equivalent inductance (L) for inductors in parallel is:

$$\boxed{L_{eq} = \left(\sum_n \left(\frac{1}{L_n}\right)\right)^{-1}}$$

Response of First Order RL Circuit

An **RL Circuit** is a circuit which contains resistors and inductors. The basic example is the following:



We want to be able to determine $v_L(t)$ and $i_L(t)$. The first thing to note is that:

$$i_s = i_R = i_L$$

.. and that:

$$v_L = L\frac{di_L}{dt}$$

... using *Ohm's Law* on the resistor:

$$v_R = i_L R$$

We can then do a single KVL loop around the circuit above, giving us:

$$-V_s + v_R + v_L = 0$$

$$-V_s + i_L R + L\frac{di_L}{dt} = 0$$

This is a first order differential equation in $i_L(t)$, and it's solution is:

$$i_L(t) = \frac{V_s}{R} + \left[(i_L(t_0) - \frac{V_s}{R})e^{\frac{-(t-t_0)}{L/R}}\right]$$

... which is also commonly written in the form:

$$\boxed{i_L(t) = \frac{V_s}{R}\left(1 - e^{\frac{-(t-t_0)}{L/R}}\right) + i_L(t_0)e^{\frac{-(t-t_0)}{L/R}}}$$

Similar terminology is used as with RC circuits, the first term is called the **source excitation response** while the second term is called the **natural response**. When solving RL circuits:

$$\tau = \frac{L}{R}$$

... which is also called the time constant like it was in RC circuits.

We can also use this equation to solve for $v_L(t)$ using:

$$v_L(t) = L\frac{di_L}{dt}$$

... which gives us:

$$\boxed{v_L(t) = V_s e^{\frac{-(t-t_0)}{\tau}} - R_{eq} i_L(t_0) e^{\frac{-(t-t_0)}{\tau}}}$$

Steps to Solve a First Order RL Circuit ($\mathbf{t}_0 = 0$)

1. Draw the circuit with initial position of the switch. (Note that a switch is usually used to control the state of the inductor).
2. Calculate the current through the inductor ($i_L(0^-)$), noting that when the inductor is fully charged it acts as a short circuit.
3. $i_L(0^-) = i_L(0^+)$
4. Draw the circuit in the final position of the switch.
5. Calculate $i_L(\infty)$ which will occur when the inductor is fully discharged assuming there are no more complications and the circuit contains only resistors and sources otherwise.
6. Calculate $\tau = \frac{L}{R}$, where R_{eq} is the total resistance seen from the perspective of the inductor with all independent sources turned off.
7. Substitute all these values into the following equation:

$$i_L(t) = i_L(\infty) + \left[i_L(0^+) - i_L(\infty)\right] e^{\frac{-t}{\tau}}$$

Chapter 9: Sinusoids and Phasors

Sinusoids

This chapter beings AC circuit analysis. A time variant voltage can be described generally by the following equation:

$$v(t) = V_m \sin(\omega t + \phi)$$

A **sinusoid** is a voltage described by an equation such as the one above. A sinusoid is said to be **leading** is it's period begins θ radians before another, and a sinusoid is said to be **lagging** if it's period begins θ radians after another.

This is called the **Time-Domain** representation of a time variable voltage. It contains information about its amplitude, frequency, and phase. The following identities are typically useful when considering the time domain:

$$\sin(\theta \pm \pi) = -\sin(\theta)$$

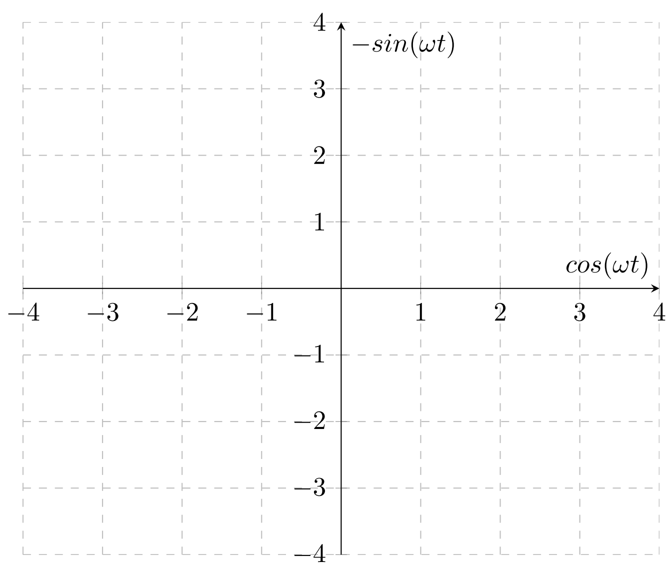
$$\cos(\theta \pm \pi) = -cos(\theta)$$

$$\sin(\theta \pm \frac{\pi}{2}) = \pm\cos(\theta)$$

$$\cos(\theta \pm \frac{\pi}{2}) = \mp\sin(\theta)$$

It is common to use the following plot to work with sinusoids with the same frequency:

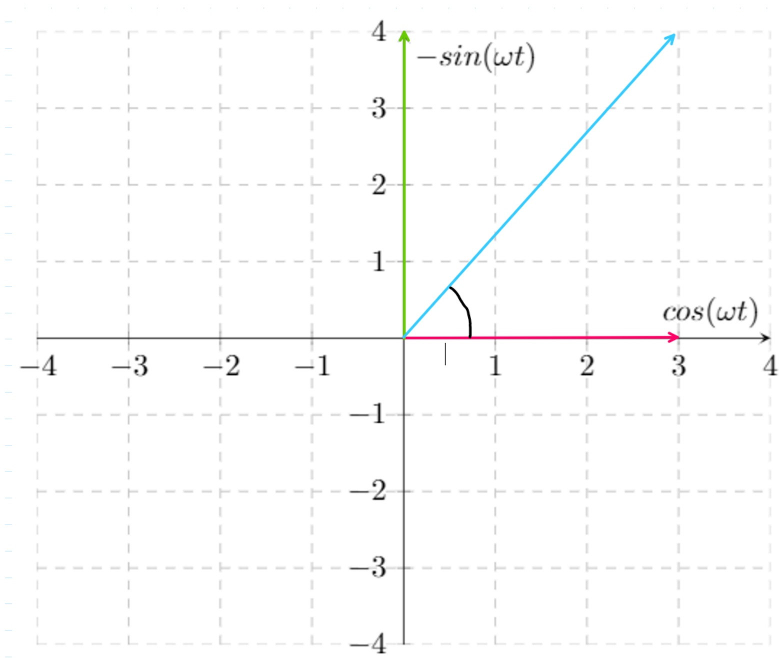
Axis used for Sinusoids



So given some expression:

$$v(t) = 3\cos(\omega t) - 4\sin(\omega t)$$

It can be expression on the grid by considering the $+x$ axis to have a basis vector of $cos(\omega t)$, and the $+y$ axis to have a basis vector of $-\sin(\omega t)$.



Where the resultant vector has a magnitude of 5 and an angle of $\theta = 53.1°$... therefore we can say that:

$$v(t) = 3\cos(\omega t) - 4\sin(\omega t) = 5\cos(\omega t + 53.1°)$$

Phasors

A phasor is a way to express a time variable quantity in a single complex number, more specifically it is a way to express the amplitude and phase of a sinusoid. Note that this course uses the symbol $\hat{\jmath}$ to denote $\sqrt{-1}$. This is called the **phasor domain** or **frequency domain** of the sinusoid, while before we were considering the time domain of the sinusoid. The *phasor domain* of the sinusoid does not contain information about the frequency of the sinusoid. This is however usually not a problem since most common electrical voltages have the same frequency around the world. The most important part of the phasor is it's magnitude and direction.

The word *phasor* comes from a combination of phase and vector, the initial position of a phasor describes the phase of the sinusoid, and it's magnitude is related to the amplitude of the sinusoid.

Recall there are three ways to describe a complex number:

$$\text{Rectangular form: } x + y\hat{\jmath}$$

$$\text{Polar form: } r\angle\phi$$

$$\text{Exponential form: } re^{\hat{\jmath}\phi}$$

... where:

$$r = \sqrt{x^2 + y^2}$$

$$\phi = \arctan\left(\frac{y}{x}\right)$$

... also recall **Euler's Formula**:

$$re^{\hat{\jmath}\phi} = r(\cos(\phi) + \hat{\jmath}\sin(\phi))$$

Also recall the **mathematical operations relating to complex numbers** for some general $z_1 = x_1 + y_1\hat{\jmath} = r_1\angle\phi_1$ and $z_2 = x_2 + y_2\hat{\jmath} = r_2\angle\phi_2$:

$$z_1 \pm z_2 = (x_1 \pm x_2) + (y_1 \pm y_2)\hat{\jmath}$$

$$z_1 \cdot z_2 = r_1 \cdot r_2\angle(\phi_1 + \phi_2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\phi_1 - \phi_2)$$

Deriving the Phasor

To begin, say we have some sinusoid in the following form (time-domain form):

$$v(t) = V_m \cos(\omega t + \phi)$$

Notice that:

$$V_m e^{\hat{\jmath}(\omega t + \phi)} = V_m \cdot \cos(\omega t + \phi) + V_m \cdot \hat{\jmath}\sin(\omega t + \phi))$$

... or:

$$\Re[V_m e^{\hat{\jmath}(\omega t + \phi)}] = V_m \cdot \cos(\omega t + \phi) = v(t)$$

Which tells us that:

$$v(t) = \Re[V_m e^{\hat{\jmath}(\omega t + \phi)}]$$

$$v(t) = \Re[V_m e^{\hat{\jmath}\omega t} e^{\hat{\jmath}\phi}] = \Re[V_m e^{\hat{\jmath}\phi} e^{\hat{\jmath}\omega t}]$$

... now we declare the following:

$$\bar{V} = V_m e^{\hat{\jmath}\phi}$$

$$v(t) = \Re[\bar{V} e^{\hat{\jmath}\omega t}]$$

$\bar{V}$ is called *V phasor*, notice that it does not depend on time. This is the phasor representation of $v(t)$:

$$\bar{V} = V_m e^{\hat{\jmath}\phi} = V_m \angle \phi$$

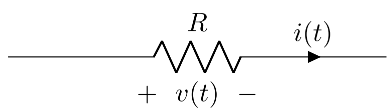
In general:

$$v(t) = V_m cos(\omega t + \phi) \implies \bar{V} = V_m \angle \phi$$

Phasors are very useful because they will eventually allow us to use DC techniques (Nodal, Mesh, Superposition, Thevenin, etc.) to solve AC circuits with a fixed voltage. Notice also that $v(t) \neq \bar{V}$, but $v(t) = \Re[\bar{V} e^{\hat{\jmath}\omega t}]$. The voltage is always time dependant, the rotating phasor is equal to the voltage, however the phasor in it's initial position is what we consider most of the time in this section. More on rotating phasors will be explored later on.

Steady State AC Analysis

For a resistor with the following properties:



We can define the current through the resistor to be in the form:

$$i(t) = I_m \cos(\omega t + \phi) \implies \bar{I} = I_m \angle \phi$$

We know that for any time t, Ohm's Law is true:

$$v(t) = R \cdot i(t)$$

$$v(t) = R \cdot I_m \cos(\omega t + \phi)$$

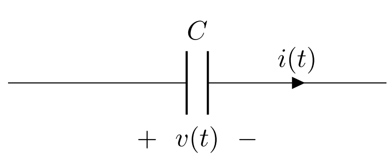
... say $V_m = R \cdot I_m$, then:

$$v(t) = V_m \cos(\omega t + \phi) \implies \bar{V} = V_m \angle \phi$$

Notice that over a resistor the current and the voltage are in phase, this will not necessarily be true for other components. This property is expressed in the following equation:

$$\bar{V} = R\bar{I}$$

For a capacitor with the following properties:



We can define the voltage through the capacitor to be in the form:

$$v(t) = V_m \cos(\omega t + \phi) \implies \bar{V} = V_m \angle \phi$$

We know that over a capacitor the following is true:

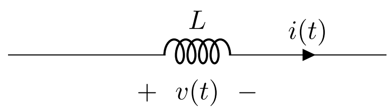
$$i(t) = C\frac{dv}{dt}$$

$$i(t) = C \cdot V_m \cdot \omega[-\sin(\omega t + \phi)]$$

... say $I_m = C \cdot V_m \cdot \omega$, then:

$$i(t) = I_m \cos(\omega t + \phi + 90^\circ) \implies \bar{I} = I_m \angle(\phi + 90^\circ)$$

Notice that this means $\bar{I}$ leads $\bar{V}$ by 90° in a perfect capacitor.
For an inductor with the following properties:



We can define the current through the inductor to be in the form:

$$i(t) = I_m \cos(\omega t + \phi) \implies \bar{I} = I_m \angle \phi$$

We know that over an inductor the following is true:

$$v(t) = L\frac{dv}{dt}$$

$$v(t) = \omega \cdot L \cdot I_m \cos(\omega t + \phi + 90^\circ)$$

... say $V_m = \omega \cdot L \cdot I_m$, then:

$$v(t) = V_m \cos(\omega t + \phi + 90^\circ) \implies \bar{V}\angle(\phi + 90^\circ)$$

The phase relationship between $\bar{I}$ and $\bar{V}$ is that $\bar{V}$ is leading $\bar{I}$ by 90°.

Impedance

Impedance is very similar to resistance, except it is frequency dependant. In order to calculate impedance you must know what frequency ω is being used to power the AC circuit. **Impedance** (Z) is defined as:

$$Z = \frac{\bar{V}}{\bar{I}}$$

Which can be applied to resistors, capacitors, and inductors using complex number arithmetic to give us:

$$\boxed{Z_R = \frac{\bar{V}}{\bar{I}} = R}$$

$$\boxed{Z_C = \frac{\bar{V}}{\bar{I}} = \frac{1}{\jmath\omega C}}$$

$$\boxed{Z_L = \frac{\bar{V}}{\bar{I}} = \jmath\omega L}$$

Both the capacitor and the inductor's impedance depend on the frequency, this is why impedance is called *frequency dependant*. Combining this with the previous section we can get the following phasor i-v characteristics of the three passive components:

$$\bar{V}_R = R\bar{I}$$

$$\bar{V}_C = \frac{\bar{I}}{\jmath\omega C}$$

$$\bar{V}_L = \jmath\omega L\bar{I}$$

Note that the following relationship is true for any of the three passive components, with the correct impedance value:

$$\bar{V} = Z\bar{I}$$

If you evaluate the formulas for impedance, we can see that there are two extremes: DC ($\omega = 0$) and high frequencies ($\omega \to \infty$). The **capacitor** acts as an open circuit to DC and a short circuit to high frequencies. The **inductor** acts as a short circuit to DC and an open circuit to high frequencies.

Impedance can be described by a complex number in rectangular form as follows:

$$Z = R \pm \jmath X$$

... where $R = \Re[Z]$ is called the resistance, and $X = \Im[Z]$ is called the reactance. Positive reactance is associated with inductance, and negative reactance is associated with capacitance. In general:

$$Z = R \pm \jmath X = |Z| \angle\theta$$

Impedance, reactance, and resistance are all measured in ohms (Ω). **Admittance** (Y) is the reciprocal of impedance measured in siemens (S) and is sometimes used, as in:

$$Y = \frac{1}{Z} = \frac{\bar{I}}{\bar{V}}$$

Admittance can be written in the form:

$$Y = G + \jmath B$$

... where $G = \Re[Y]$ is conductance, and $B = \Im[Y]$ is susceptance, all measured in siemens. The relationship between $R, X, G, \&\ B$ is described by the following:

$$G = \frac{R}{R^2 + X^2}$$

$$B = -\frac{X}{R^2 + X^2}$$

A very useful thing about *impedances* is that since they are all measured in ohms, they can be combined like resistors would be, allowing for things like easier nodal analysis, and current/voltage division.

Chapter 10: AC Circuit Analysis

Solving AC Circuits

The following are the steps to solve a simple AC circuit:

1. Fix the frequency to some constant value.
2. Convert all components from time-domain to phasor-domain:
   - For a voltage source:
     $$v(t) = V_m \cos(\omega t + \phi) \implies \bar{V} = V_m \angle \phi$$
   - For a current source:
     $$i(t) = I_m \cos(\omega t + \phi) \implies \bar{I} = I_m \angle \phi$$
   - For a resistor:
     $$R(\Omega) \implies R(\Omega)$$
   - For a capacitor:
     $$C(F) \implies \frac{1}{\jmath \omega C}(\Omega)$$
   - For an inductor:
     $$L(H) \implies \jmath \omega L(\Omega)$$
3. Analyze circuit in phasor-domain using any DC circuit technique.
   - Systems of linear equations containing complex numbers can be solved using *Cramer's Rule.*
4. At the same fixed frequency, convert all results to time-domain form.

Chapter 11: AC Power Analysis

In practical circuit analysis, power is usually the most important variable to understand and measure, and so the final chapter of this course is dedicated to this.

Instantaneous Power (p) absorbed by an element is the product of the instantaneous voltage and the instantaneous current thought the element being measured, as in:

$$\boxed{p(t) = v(t)i(t)}$$

It is the rate at which an element absorbs energy, measured in Watts. This is an instantaneous quantity.

Given the following voltage and current setup:

$$v(t) = V_m \cos(\omega t + \phi_v)$$

$$i(t) = I_m \cos(\omega t + \phi_i)$$

... then instantaneous power can be sinusoidally defined by:

$$p(t) = \frac{1}{2}V_m I_m \cos(\phi_v - \phi_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \phi_v + \phi_i)$$

This formula shows us that power has two distinct components, a time independent part which is the first term, and then a time dependant part with a frequency of 2ω which is the second term.

When this representation of $p(t)$ is graphed, it is sinusoidal. The positive intervals mean that power is absorbed by the circuit (like resistors). The negative intervals mean that power is being transferred from the circuit to the source (like capacitors and inductors).

Instantaneous Power is very difficult to measure, and so watt-meters measure **average power** (P_{ave}) which is the average of all the instantaneous power over one period T, mathematically that is:

$$P_{ave} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{2}V_m I_m cos(\phi_v - \phi_i)$$

Importantly the second part of the definition does not at all depend on time. In phasor domain that is:

$$P_{ave} = \frac{1}{2}\Re[\bar{V}\bar{I}^*] = \frac{1}{2}V_m I_m \cos(\phi_v - \phi_i)$$

A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power. *Average Power* can also be called active power or real power, as it is the real part of the complex number above.

Reactive Power (Q) is defined as the complex part of the above expression, and it is the energy exchange between the source and the reactive part of the load (R or L), mathematically:

$$Q = \frac{1}{2}\Im[\bar{V}I^*] = \frac{1}{2}V_m I_m \sin(\phi_v - \phi_i)$$

Reactive power has units of VAR.

Measured Quantities

Measuring devices like voltmeters and ammeters measure *effective values* or *rms value*, which are equivalent terms. Tools measure the effectiveness of a voltage or current at delivering power to a resistive load. The *rms* value which stands for Root Mean Square is the DC value that would deliver the same average power as the AC circuit, and is defined as:

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T (v(t))^2 dt}$$

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T (i(t))^2 dt}$$

... which is true for any signal. If $i(t)$ is a sinusoid, then (similar reasoning for V_{rms}):

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

$$I_{rms} = \frac{I_m}{\sqrt{2}}$$

Average or real power can then be written in terms of *rms* values:

$$P = V_{rms} I_{rms} \cos(\phi_v - \phi_i)$$

... and average power absorbed by a resistor is:

$$P = I_{rms}^2 R = \frac{V_{rms}^2}{R}$$

Apparent Power

Apparent power (S) is the product of the two *rms* values, as in:

$$P = V_{rms} I_{rms} \cos(\phi_v - \phi_i) = S \cos(\phi_v - \phi_i)$$

... and the cosine is the *power factor* (pf). It is called apparent power since it is apparent that power should be that product, however is it measured in VA (Volt-Amperes) as to distinguish it from Watts. The power factor can be described by:

$$pf = \frac{P}{S} = \cos(\phi_v - \phi_i) = \cos(\phi_S)$$

The power factor is the cosine of the phase difference between the voltage and the current, which is equivalently the cosine of the angle of the load impedance. This can be thought of as the factor by which the apparent power must be multiplied by to obtain the real power, which is a number from 0 to 1. For a purely resistive load that $pf = 1$ since the two signals are in phase. If the power factor is 1 then the system has pure power.

Complex Power

Complex power contains all the information pertaining to the power absorbed by a given load.

$$\bar{S} = \frac{1}{2}\bar{V}\bar{I}^* = \bar{V}_{rms}\bar{I}^*_{rms}$$

Where:

$$\bar{V}_{rms} = \frac{\bar{V}}{\sqrt{2}} = \bar{V}_{rms}\angle\phi_v$$

$$\bar{I}_{rms} = \frac{\bar{I}}{\sqrt{2}} = \bar{I}_{rms}\angle\phi_i$$

We know now then that:

$$\bar{S} = V_{rms}I_{rms}\cos(\phi_v - \phi_i) + \jmath V_{rms}I_{rms}\sin(\phi_v - \phi_i)$$

Since the magnitude of complex power is apparent power, it is also measured in VA, and the angle of the complex power is the power factor angle. We also know that:

$$\bar{V}_{rms} = \bar{Z}\bar{I}_{rms}$$

... so therefore we know:

$$\bar{S} = \left|I_{rms}\right|^2 \bar{Z} = \frac{\left|V_{rms}\right|^2}{\bar{Z}^*} = \bar{V}_{rms}\bar{I}^*_{rms}$$

Recall average power and reactive power, those were defined above in terms of complex power without using the term, formally they are defined as:

$$P_{ave} = \Re[\bar{S}] = V_{rms}I_{rms}\cos(\phi_v - \phi_i) = I^2_{rms}R$$

$$Q = \Im[\bar{S}] = V_{rms}I_{rms}\sin(\phi_v - \phi_i) = I^2_{rms}X$$

Real power P is the only useful power, which is the power dissipated by the load. Q is measured in volt ampere reactive (VAR), which is transferred back and forth between the load and the source.

The following equations are a recap of all we need to know from AC Power:

$$\text{Complex Power: } \bar{S} = P + \jmath Q = \bar{V}_{rms}\bar{I}^*_{rms} = \left|\bar{V}_{rms}\right|\left|\bar{I}_{rms}\right|\angle\phi_v - \phi_i$$

$$\text{Apparent Power: } S = \left|\bar{S}\right| = \left|\bar{V}_{rms}\right|\left|\bar{I}_{rms}\right| = \sqrt{P^2 + Q^2}$$

$$\text{Real Power: } P = \Re[\bar{S}] = S\cos(\phi_v - \phi_i)$$

$$\text{Reactive Power: } Q = \Im[\bar{S}] = S\sin(\phi_v - \phi_i)$$

$$\text{Power Factor: } \frac{P}{S} = \cos(\phi_v - \phi_i) = \cos(\phi_S)$$

Finally to conclude this section, note that all forms of AC power are conservative, and that the power from the source equals the respective sums of the complex, real, and reactive powers of the individual loads.